ON CHARACTER DEGREES OF FINITE GROUPS AND SOME ASSOCIATED GRAPHS

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Abstract

We present some interesting results that reflect the strong interplay between the structure of a finite group G and the properties of some graphs associated with its character degree set.

Preliminaries in Group Theory

Definition:
Let G be a finite group and V be a vector space over $\mathbb{C}$, such that $dim_{\mathbb{C}}(V) = n$. A representation of G in V of degree n is a homomorphism of groups $\rho : G \longrightarrow GL(V) \cong GL(n, \mathbb{C})$.
It is called irreducible if and only if V has no nontrivial invariant subspaces.
The character of G associated with ρ is the mapping $\chi: G \longrightarrow \mathbb{C}$ where $\chi(g) = tr(\rho(g))$. A character of G is called irreducible if and only if ρ is irreducible. The set of all irreducible characters of G is denoted by $Irr(G)$.
If $H \trianglelefteq G$, $\theta \in Irr(H)$ and $g \in G$, we define $\theta^g : H \longrightarrow \mathbb{C}$ by $\theta^g(h) = \theta(ghg^{-1})$, we have $\theta^g \in Irr(H)$ and

$$I_G(\theta) = \{g \in G \colon \theta^g = \theta\}.$$

Definition:
A subnormal series of G is a finite chain of subgroups of G:

$$G_0 = \{1_G\} \leq G_1 \leq \cdots \leq G_r = G$$

where $G_{i-1} \trianglelefteq G_i$ for every $1 \leq i \leq r$.
If G_i/G_{i-1} is abelian for every $1 \leq i \leq r$, then G is solvable and the length of the shortest subnormal series of G verifying this property is called the derived length of G and it is denoted by $dl(G)$.

Definition:
A solvable radical of a group G is the largest solvable normal subgroup of G.

Definition:
A Hall subgroup H of a group G is a subgroup whose order is relatively prime with its index.

Definition:
A group G is said to be simple if and only if it has no proper normal subgroups.

Definition:
A group G is an almost simple group if and only if there exists a simple nonabelian group S such that $S \leq G \leq Aut(S)$. In this case, we say that G is an almost simple group with socle S.

Preliminaries in Graph Theory

Definitions:

1. A graph Ω is an ordered pair $(V(\Omega), E(\Omega))$, where $V(\Omega)$ is a vertex set, $E(\Omega)$ is an edge set and each edge is associated with two vertices.
2. A walk on Ω is an alternating sequence of vertices and edges: $v_0, e_1, v_1, \cdots, e_k, v_k$ such that v_{i-1} and v_i are the endpoints of the edge e_i for all $1 \leq i \leq k$. The length of a walk is defined to be the number of its edges.
3. A path of length n, P_n, is a walk where all vertices are distinct.
4. A cycle of length n, C_n, is a walk where $v_0 = v_k$ and all other vertices are distinct.
5. A graph Ω is connected if for all $u \neq v \in V(\Omega)$ there exists a $u, v - path$ (a path whose endpoints are u and v). Otherwise we say that Ω is disconnected. In this case, we define the connected components of Ω to be its maximal connected subgraphs, and we denote by $n(\Omega)$ the number of such components. If Ω is connected, then, $diam(\Omega) = Max\{d_{\Omega}(u, v) \colon u, v \in \Omega\}$, where $d_{\Omega}(u, v)$ is the length of the shortest $u, v - path$.

Basic Definitions and Theorems

Definition:
Let G be a finite group. Consider the set of irreducible complex characters of G, $Irr(G)$, and define $cd(G) = \{\chi(1) \colon \chi \in Irr(G)\}$. Let $\rho(G)$ be the set of all primes which divide some character degrees of G.

- The prime degree graph, $\Delta(G)$, is the graph whose vertex set is $\rho(G)$ and there is an edge between two primes p and q in $\rho(G)$ if and only if pq divides some degree in $cd(G)$.
- The common divisor degree graph, $\Gamma(G)$, is the graph whose vertex set is $cd(G)^* = cd(G) \setminus \{1\}$ and there is an edge between two characters m and n if and only if $\gcd(m, n) \neq 1$.
- The bipartite divisor graph, $B(G)$, is the graph whose vertex set is $\rho(G) \cup cd(G)^*$ and there is an edge between $p \in \rho(G)$ and $m \in cd(G)^*$ if and only if $p|m$.

These graphs are strongly related graphs, in particular, they have the same number of connected components.

Theorem:
For any finite group G, we have $n(\Delta(G)) \leq 3$.
If G is solvable, then $n(\Delta(G)) \leq 2$.

Theorem:
There is no solvable group G whose $\Delta(G)$ is a P_3 or a C_4.

Theorem:
There is no nonsolvable group G whose $\Delta(G)$ is a P_3.

Theorem:
Let G be a nonsolvable group such that $|cd(G)^*| = 4$. Then $\Gamma(G)$ is one of the graphs listed in Figures 1 and 2.

Theorem:
Let G be a finite group such that $\Delta(G)$ has no triangles (C_3's). Then,

$$|\rho(G)| \leq 5.$$

Theorem:
Let G be a finite group such that $\Delta(G)$ has five vertices and no triangles, then the following hold:

1. If $\Delta(G)$ is disconnected, then $G \cong PSL(2, 2^f) \times A$, where A is abelian, $|\pi(2^f \pm 1)| = 2$ and $\Delta(G)$ is the second graph in Figure 3.
2. If $\Delta(G)$ is connected, then $G = H \times K$, where $H \cong A_5$ or $PSL(2,8)$, K is a solvable group such that $\Delta(K)$ has exactly two vertices and two connected components and $\rho(H) \cap \rho(K) = \emptyset$. Furthermore, $\Delta(G)$ is the first graph in Figure 3.

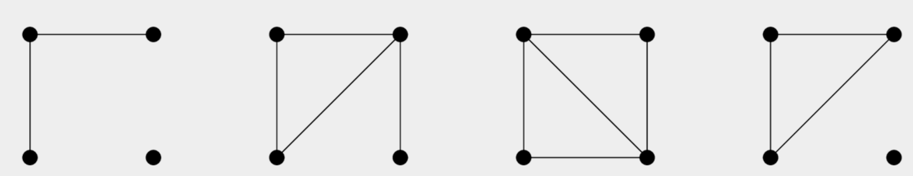

Figure 1: $\Gamma(G)$ when G is nonsolvable and $|cd(G)^*| = 4$.

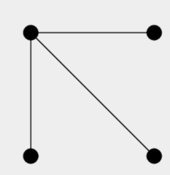

Figure 2: Unknown graph.

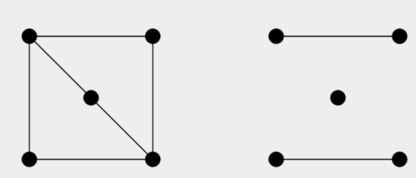

Figure 3: Graphs of finite groups having five vertices and no triangles.

Result 1

Theorem:
Let G be a finite group. Then $diam(B(G)) \leq 7$ and this bound is sharp.

Theorem:
Let G be a finite group such that $B(G) = P_n$ for some positive integer n. Then, $n \leq 6$, G is solvable and $dl(G) \leq 5$.

Example:
Consider $G = S_3 \times A_4$. We have:
$cd(S_3) = \{1,2\}, cd(A_4) = \{1,3\}$ and $cd(G) = \{1,2,3,6\}$.
Moreover, G is solvable as S_3 and A_4 are so.

Result 2

Lemma:
Let S be a nonabelian simple group. Then $B(S)$ is disconnected and all its connected components are paths if and only if S is isomorphic to one of the following groups:

i. $PSL(2, 2^n)$ where $|\pi(2^n \pm 1)| \leq 2$;

ii. $PSL(2, p^n)$ where p is an odd prime and $|\pi(p^n \pm 1)| \leq 2$.

Lemma:
If G is a finite group whose $B(G)$ is a union of paths and $|\rho(G)| = 5$, then $G \cong PSL(2, 2^n) \times A$, where A is abelian and $|\pi(2^n \pm 1)| = 2$.

Theorem:
Let G be a finite nonsolvabe group and let N be the solvable radical of G. If $B(G)$ is a union of paths, then $B(G)$ is disconnected and there exists a normal subgroup M of G such that G/N is an almost simple group with socle M/N. Furthermore, $\rho(G) = \rho(M)$ and we have one of the following cases:

1) If $n(B(G)) = 2$, then $|cd(G)| = 5$ or $|cd(G)| = 4$, $G/N \in \{PGL(2, q), M_{10}\}$, for $q > 3$ odd, and either $cd(G) = \{1, q-1, q, q+1\}$ or $cd(G) = \{1, 9, 10, 16\}$.
Let C_1 and C_2 be the connected components of $B(G)$. Then C_1 is a path of length one and $C_2 \cong P_n$ where $n \in \{|\rho(G)|, |\rho(G)| + 1\}$.

2) If $n(B(G)) = 3$, then $G \cong PSL(2, 2^n) \times A$, where A is an abelian group and $n \geq 2$.

Example:
Let $G = PSL(2,25)$. Then $cd(G) = \{1, 13, 24, 25, 26\}$.
Thus $B(G)$ is the union of the following two paths:

$$C_1 : 5 - 25, \text{ and}$$

$$C_2 : \quad 13 - 13 - 26 - 2 - 24 - 3.$$

Result 3

Lemma:
Let G be a finite group whose $B(G)$ is a cycle of length $n \geq 6$. Then, $\Delta(G)$ and $\Gamma(G)$ are both cycles.

Theorem:
If G is a finite group whose $B(G)$ is a cycle of length n. Then,

$$n \in \{4, 6\}$$

Corollary:
If G is a finite group whose $B(G)$ is a cycle, then G is solvable and

$$dl(G) \leq |cd(G)| \leq 4.$$

Theorem:
Let G be a finite group such that $B(G)$ is a cycle of length four. Then there exists a normal abelian Hall subgroup N of G such that

$$cd(G) = \{|G \colon I_G(\lambda)| \colon \lambda \in Irr(N)\}.$$

Example:
Consider the two nonabelian groups among the sixty six groups of order 588. The bipartite divisor graph of these two groups is C_4, precisely, these groups have $\{1, 6, 12\}$ as their irreducible character degree set.
Remark that any group of order 588 is solvable.

Interesting Questions

Question 1:
Is there any solvable group G whose $B(G)$ is a path of length four such that $|cd(G)| = 4$?

Question 2:
Is there any solvable group G whose $B(G)$ is a P_5 or a P_6?

References

[1] Hafezieh R., (2017), "Bipartite divisor graph for the set of irreducible character degrees", International Journal of Group Theory, 6 (4), 41-51.

[2] LiGuo H., GuoHua Q., (2015), "Graphs of nonsolvable groups with four degree-vertices", Sci. China Math., 58 (6), 1305–1310.

[3] Lewis M., White D., (2013), "Four-vertex degree graphs of nonsolvable groups", J. Algebra, 378, 1–11.

[4] Lewis M., (2008), "An overview of graphs associated with character degrees and conjugacy class sizes in finite groups", Rocky Mountain J. Math., 38 (1), 175–211.

[5] Tong-Viet H. P., (2013), "Groups whose prime graph has no triangles", J. Algebra, 378, 196–206.

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